

# Three-Point Density Correlation Functions in the Fractional Quantum Hall Regime

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## Abstract

In this paper we consider the three-particle density correlation function for a fractional quantum Hall liquid. The study of this object is motivated by recent experimental studies of fractional quantum Hall systems using inelastic light scattering and phonon absorption techniques. Symmetry properties of the correlation function are noted. An exact sum-rule is derived which this quantity must obey. This sum-rule is used to assess the convolution approximation that has been used to estimate the matrix elements for such experiments. PACS Numbers: 73.40.Hm, 73.20.Mf, 72.10.Di

## 1 Introduction

The fractional quantum Hall effect has been one of the key topics in condensed matter physics for the last 15 years. While much of the experimental work on this phenomenon has been focussed on electrical transport measurements, recently attention has turned to the development of complementary spectroscopic techniques. In the case of both phonon[1, 2] and inelastic light scattering[3, 4] experiments the spectroscopic probe couples to the electronic density of the system and so naturally interacts with its collective density fluctuations. The theory of these collective modes was first developed by Girvin, MacDonald and Platzman (GMP)[5] in a manner analogous to that used by Feynman for the phonon and roton modes of superfluid helium. Unlike the case of liquid helium, there is no gapless phonon-like behaviour at small wave vector; the collective modes of the fractional quantum Hall systems are gapped at all wavelengths: a consequence of the incompressibility of these states. However, in common with liquid helium, the collective mode dispersion does have a well defined minimum, hence the quanta of collective excitations of the fractional quantum Hall states are referred to as magnetorotons.

One can envision two types of process involved in both the phonon and light scattering experiments. Firstly a magnetoroton can be created when the

electron liquid absorbs one of the probe quanta (phonon or photon), we refer to this as type A absorption. Secondly, an existing magnetoroton can be scattered into a different state by the absorption of a probe quantum, this we call a type B process. At zero temperature the electron liquid will be in its ground state (well described by Laughlin's wave function[6]) and there will be no magnetorotons present, hence only the type A process can occur. Because the magnetoroton dispersion is gapped at all wave vectors there is a threshold frequency for this process: no probe quantum can be absorbed whose energy is less than the minimum gap,  $\Delta^*$ . Type B processes will occur at any non-zero temperature and have no threshold: in principal, arbitrarily small energies can be transferred to the electron liquid by these processes. The theory of the Type A processes has been discussed by He and Platzman [7] for the inelastic light-scattering case and by our group [8] for the phonon case. These type B processes will determine the leading finite-temperature corrections to the zero temperature theory outlined in [8]. The theory of Type B processes is less well developed but potentially rather important for the understanding of the phonon experiments because these typically use pulses containing a black body distribution of phonon energies characterized by a pulse temperature  $T_\phi < \Delta^*$  and use the temperature of the electron liquid itself to monitor the rate of phonon absorption. At an arbitrary electron temperature there will be competition between rare type A processes in which high energy phonons are absorbed and common type B processes in which low energy phonons are absorbed. In recent time-resolved experiments[2] the change in electron temperature is recorded over the period of time in which the pulse is in contact with the electrons. At early times the electron liquid is at very low temperatures and so type A processes will dominate the energy transfer and hence the rate of change of electron temperature. At later times the electron temperature will be higher, because of all the earlier type A absorptions and type B processes will become comparable in determining the energy transfer rate.

The purpose of this paper is to derive a sum rule obeyed by the matrix elements which control the rates for the type B processes and to use this to test the best existing approximation scheme for this object: the colvolution approximation developed for studying the corresponding quantities for superfluid Helium-4 [12]. As discussed in [11], these matrix elements are also important for the understanding of the effects of disorder on the form of the magnetoroton dispersion. In all of the following, natural units in which  $l_c = \sqrt{\hbar/eB} = 1$  and  $e^2/4\pi\epsilon_0\kappa l_c = 1$  (where  $\kappa$  is the bulk dielectric constant of the material in which the 2des is formed) are used.

## 2 Magnetorotons

Girvin, MacDonald and Platzman[5] supposed that the low-lying collective excitations of a fractional quantum Hall fluid could be obtained in a manner analogous to that used by Feynman in his classic works on superfluid helium. They proposed that a collective excitation with 2d wave vector  $\mathbf{q}$  would be described

by the wave function

$$|\mathbf{q}\rangle = \frac{\bar{\rho}_{\mathbf{q}} |\Omega\rangle}{\sqrt{\langle \Omega | \bar{\rho}_{-\mathbf{q}} \bar{\rho}_{\mathbf{q}} | \Omega \rangle}} \quad (1)$$

where  $|\Omega\rangle$  is the ground state of the system (for which Laughlin has provided an excellent wave function, at least for the cases where the Landau level filling  $\nu$  is of the form  $1/m$  for  $m$  odd) and  $\bar{\rho}_{\mathbf{q}}$  is an operator that is derived from the conventional density operator

$$\rho_{\mathbf{q}} = \sum_{i=1}^N e^{i\mathbf{q} \cdot \hat{\mathbf{r}}_i} \quad (2)$$

by projection onto the lowest Landau level. In the notation used by MacDonald in [9]

$$\bar{\rho}_{\mathbf{q}} = \sum_{i=1}^N B_i(\mathbf{q}) \quad (3)$$

$$B(\mathbf{q}) = \mathcal{P}_0 e^{i\mathbf{q} \cdot \mathbf{r}} \mathcal{P}_0 \quad (4)$$

where  $\mathcal{P}_0$  acts on the Hilbert space of a single electron to project out the states within the lowest (spin-polarized) Landau level. Girvin and Jach[10] investigated the properties of these projected operators and, in particular deduced that

$$B_i(\mathbf{k}) B_i(\mathbf{q}) = e^{k^* \mathbf{q}/2} B_i(\mathbf{k} + \mathbf{q}) \quad (5)$$

where  $\mathbf{k} = k_x + ik_y$ . This leads, bearing in mind that the projection operators for one particle commute with operators associated with the other, to the commutation relation for the projected density operators

$$\begin{aligned} [\bar{\rho}_{\mathbf{k}}, \bar{\rho}_{\mathbf{q}}] &= \left( e^{k^* \mathbf{q}/2} - e^{q^* \mathbf{k}/2} \right) \bar{\rho}_{\mathbf{k}+\mathbf{q}} \\ &= i\Phi(\mathbf{k}, \mathbf{q}) \bar{\rho}_{\mathbf{k}+\mathbf{q}} \end{aligned} \quad (6)$$

where

$$\begin{aligned} \Phi(\mathbf{k}, \mathbf{q}) &= -i \left( e^{k^* \mathbf{q}/2} - e^{q^* \mathbf{k}/2} \right) \\ &= e^{k \cdot \mathbf{q}/2} 2 \sin \left( \frac{1}{2} \mathbf{k} \wedge \mathbf{q} \right) . \end{aligned} \quad (7)$$

The normalization in equation 1 includes a matrix element which has the form of a projected static structure factor

$$\bar{s}(q) = \frac{1}{N} \langle \Omega | \bar{\rho}_{-\mathbf{q}} \bar{\rho}_{\mathbf{q}} | \Omega \rangle \quad (8)$$

GMP showed that this could be related to the true static structure factor

$$s(q) = \frac{1}{N} \langle \Omega | \rho_{-\mathbf{q}} \rho_{\mathbf{q}} | \Omega \rangle \quad (9)$$

via the simple relation

$$\bar{s}(q) - e^{-q^2/2} = s(q) - 1 \quad . \quad (10)$$

The magnetoroton energy can be written in the form

$$\begin{aligned} \Delta(k) &= \langle \mathbf{k} | \mathcal{H} | \mathbf{k} \rangle - \langle \Omega | \mathcal{H} | \Omega \rangle \\ &= \frac{\langle \Omega | \bar{\rho}_{-\mathbf{k}} [\mathcal{H}, \bar{\rho}_{\mathbf{k}}] | \Omega \rangle}{N \bar{s}(k)} \\ &= \frac{\bar{f}(k)}{\bar{s}(k)} \end{aligned} \quad (11)$$

and GMP showed that the projected oscillator strength,  $\bar{f}(k)$ , can be written explicitly as an integral involving  $\bar{s}$  and  $\Phi$  only. Hence, obtaining  $\bar{s}(k)$  from the Laughlin wave function, they found the dispersion relation for these excitations.

Consider now a bosonic probe quantum (such as a phonon or photon) labelled by a 3d wave vector  $\mathbf{Q}$  which couples to the 2d electrons via a hamiltonian of the form

$$H_{int} = \sum_{\mathbf{Q}} M_{\mathbf{Q}} \rho_{\mathbf{q}} (a_{\mathbf{Q}} + a_{-\mathbf{Q}}^\dagger) \quad (12)$$

where  $a_{\mathbf{Q}}$  annihilates a probe quantum with the given wave vector,  $\rho_{\mathbf{q}}$  is the electron density operator described above (un-projected) and  $M_{\mathbf{Q}}$  is some coupling function. For the detailed forms of  $M_{\mathbf{Q}}$  relevant to the experimental systems see [7, 8]. The probability per unit time for a magnetoroton initially in a state  $|\mathbf{k}\rangle$  to absorb a probe quantum with wave vector  $\mathbf{Q}$  and be scattered into the state  $|\mathbf{k} + \mathbf{q}\rangle$  is given by Fermi's golden rule as

$$\tau_{\mathbf{k}, \mathbf{Q}}^{-1} = \frac{2\pi}{\hbar} |M_{\mathbf{Q}}|^2 |\langle \mathbf{k} + \mathbf{q} | \rho_{\mathbf{q}} | \mathbf{k} \rangle|^2 \delta(\Delta(|\mathbf{k} + \mathbf{q}|) - \Delta(k) - \hbar\omega_{\mathbf{Q}}) \quad (13)$$

where  $\omega_{\mathbf{Q}}$  is the energy of the probe quantum. This, as usual with lowest order perturbation theory, factorizes nicely into a part which depends on the well characterized details of the probe and the material in which the 2des is formed and a part that only involves the states of the correlated 2d electrons. The relevant matrix element in the latter is

$$\langle \mathbf{k} + \mathbf{q} | \rho_{\mathbf{q}} | \mathbf{k} \rangle = \frac{P(\mathbf{k}, \mathbf{q})}{\sqrt{\bar{s}(|\mathbf{k} + \mathbf{q}|) \bar{s}(k)}} \quad (14)$$

where we have replaced  $\rho_{\mathbf{q}}$  with its projected counterpart by virtue of the idempotence of the projection operators and we define the three-point correlation function

$$P(\mathbf{k}, \mathbf{q}) = \frac{1}{N} \langle \Omega | \bar{\rho}_{-\mathbf{k}-\mathbf{q}} \bar{\rho}_{\mathbf{q}} \bar{\rho}_{\mathbf{k}} | \Omega \rangle \quad . \quad (15)$$

### 3 Symmetry Properties of the Correlation Function

We can deduce two symmetry properties of  $P(\mathbf{k}, \mathbf{q})$  straightforwardly. Firstly we can use the fact that  $\bar{\rho}_{\mathbf{q}}^\dagger = \bar{\rho}_{-\mathbf{q}}$  to deduce that

$$\begin{aligned} \{P(\mathbf{k}, \mathbf{q})\}^* &= \left\{ \frac{1}{N} \langle \Omega | \bar{\rho}_{-\mathbf{k}-\mathbf{q}} \bar{\rho}_{\mathbf{q}} \bar{\rho}_{\mathbf{k}} | \Omega \rangle \right\}^* \\ &= \frac{1}{N} \langle \Omega | \bar{\rho}_{-\mathbf{k}} \bar{\rho}_{-\mathbf{q}} \bar{\rho}_{\mathbf{k}+\mathbf{q}} | \Omega \rangle \\ &= P(\mathbf{k} + \mathbf{q}, -\mathbf{q}) \quad . \end{aligned} \quad (16)$$

Secondly we can use the commutation relation for the projected density operators, as derived by Girvin and Jach[10] to simplify

$$\begin{aligned} P(\mathbf{k}, \mathbf{q}) - P(\mathbf{q}, \mathbf{k}) &= \frac{1}{N} \langle \Omega | \bar{\rho}_{-\mathbf{k}-\mathbf{q}} [\bar{\rho}_{\mathbf{q}}, \bar{\rho}_{\mathbf{k}}] | \Omega \rangle \\ &= i\Phi(\mathbf{q}, \mathbf{k}) \frac{1}{N} \langle \Omega | \bar{\rho}_{-\mathbf{k}-\mathbf{q}} \bar{\rho}_{\mathbf{k}+\mathbf{q}} | \Omega \rangle \\ &= i\Phi(\mathbf{q}, \mathbf{k}) \bar{s}(|\mathbf{k} + \mathbf{q}|) \quad . \end{aligned} \quad (17)$$

## 4 Derivation of the Sum Rules

### 4.1 Structure Factor Sum-Rule

In order to show the basic idea of the method we will begin with a simple case and prove a sum-rule for the projected structure factor itself. From the work of GMP [5] we know that

$$\bar{s}(q) - e^{-q^2/2} = s(q) - 1 \quad .$$

In first quantized notation we have that, for a system of  $N$  particles

$$\begin{aligned} s(q) &= \frac{1}{N} \langle \Omega | \left( \sum_{i=1}^N e^{-i\mathbf{q} \cdot \hat{\mathbf{r}}_i} \right) \left( \sum_{j=1}^N e^{i\mathbf{q} \cdot \hat{\mathbf{r}}_j} \right) | \Omega \rangle \\ &= 1 + \frac{1}{N} \sum_{i \neq j} \langle \Omega | e^{-i\mathbf{q} \cdot (\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j)} | \Omega \rangle \end{aligned}$$

Hence we have that

$$\int (s(\mathbf{q}) - 1) d^2\mathbf{q} = \frac{1}{N} \sum_{i \neq j} \langle \Omega | \int e^{-i\mathbf{q} \cdot (\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j)} d^2\mathbf{q} | \Omega \rangle$$

Now

$$\int e^{-i\mathbf{q} \cdot (\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j)} d^2\mathbf{q} = \delta(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j)$$

so that

$$\int (s(\mathbf{q}) - 1) d^2\mathbf{q} = \frac{1}{N} \sum_{i \neq j} \langle \Omega | \delta(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j) | \Omega \rangle \quad .$$

Inserting a representation of unity in terms of the complete set of  $N$ -particle position eigenstates  $|\{\mathbf{r}_i\}\rangle$  gives

$$\begin{aligned} \int (s(\mathbf{q}) - 1) d^2\mathbf{q} &= \frac{1}{N} \sum_{i \neq j} \int \left( \prod_{i=1}^N d^2\mathbf{r}_i \right) \langle \Omega | \delta(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j) | \{\mathbf{r}_i\} \rangle \langle \{\mathbf{r}_i\} | \Omega \rangle \\ &= \frac{1}{N} \sum_{i \neq j} \int \left( \prod_{i=1}^N d^2\mathbf{r}_i \right) \delta(\mathbf{r}_i - \mathbf{r}_j) \langle \Omega | \{\mathbf{r}_i\} \rangle \langle \{\mathbf{r}_i\} | \Omega \rangle \end{aligned}$$

The only configurations that could contribute to this integral are those in which two particles co-incide. Such configurations, however, have zero weight because of the Pauli exclusion principle. Hence we conclude that

$$\int (s(\mathbf{q}) - 1) d^2\mathbf{q} = 0 \quad .$$

From this we find, straightforwardly that

$$\int \bar{s}(q) d^2\mathbf{q} - 2\pi \int_0^\infty e^{-q^2/2} q dq = \int (s(q) - 1) d^2\mathbf{q} = 0$$

or

$$\int \bar{s}(q) d^2\mathbf{q} = 2\pi \quad .$$

Of course, this result can be derived more simply by noting that

$$s(\mathbf{q}) = 1 + \rho \int e^{-i\mathbf{q} \cdot \mathbf{r}} (g(r) - 1) d^2\mathbf{r} + 4\pi^2 \rho \delta^2(\mathbf{q})$$

where  $g(r)$  is the radial distribution function. Hence

$$\begin{aligned} \int (s(\mathbf{q}) - 1) d^2\mathbf{q} &= \rho \int \int e^{-i\mathbf{q} \cdot \mathbf{r}} d^2\mathbf{q} (g(r) - 1) d^2\mathbf{r} + 4\pi^2 \rho \int \delta^2(\mathbf{q}) d^2\mathbf{q} \\ &= 4\pi^2 \rho \int \delta^2(\mathbf{r}) (g(r) - 1) d^2\mathbf{r} + 4\pi^2 \rho \\ &= 4\pi^2 \rho g(0) \quad . \end{aligned}$$

The Pauli principle ensures that  $g(0) = 0$  so that we again recover our sum-rule.

## 4.2 Three Point Correlation Sum-Rule

Now let us derive our principal result: a sum rule for the three point correlation function. As shown by MacDonald et al.[11], the three point correlation function can be written in the form

$$P(\mathbf{k}, \mathbf{q}) = P_0(\mathbf{k}, \mathbf{q}) + h^{(3)}(\mathbf{k}, \mathbf{q}) \quad (18)$$

where

$$P_0(\mathbf{k}, \mathbf{q}) = e^{-\frac{1}{2}|\mathbf{q}|^2 - \frac{1}{2}\mathbf{k}^* \mathbf{q}} \left\{ e^{-\frac{1}{2}|\mathbf{k}|^2} + s(|\mathbf{k}|) - 1 \right\} \\ + e^{-\frac{1}{2}(\mathbf{k}+\mathbf{q})^* \mathbf{k}} (s(|\mathbf{q}|) - 1) + e^{\frac{1}{2}\mathbf{q}^* \mathbf{k}} (s(|\mathbf{k} + \mathbf{q}|) - 1) \quad .$$

Once again,  $\mathbf{k} = k_x + ik_y$  is the complex representation of the vector  $\mathbf{k}$  and

$$h^{(3)}(\mathbf{k}, \mathbf{q}) = \frac{1}{N} \sum_{l \neq m \neq n} \left\langle \Omega \left| e^{-i(\mathbf{k}+\mathbf{q}) \cdot \mathbf{r}_l} e^{i\mathbf{q} \cdot \mathbf{r}_m} e^{i\mathbf{k} \cdot \mathbf{r}_n} \right| \Omega \right\rangle \quad (19)$$

is the un-projected three point correlation function of the quantum liquid.

We wish to find a sum rule of the form

$$\frac{1}{\mathcal{A}} \sum_{\mathbf{q}} P(\mathbf{k}, \mathbf{q}) = F(\mathbf{k}) \quad . \quad (20)$$

The term that is likely to cause us difficulty is the un-projected three point function, however

$$\frac{1}{\mathcal{A}} \sum_{\mathbf{q}} h^{(3)}(\mathbf{k}, \mathbf{q}) = \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} \frac{1}{N} \sum_{l \neq m \neq n} \left\langle \Omega \left| e^{-i(\mathbf{k}+\mathbf{q}) \cdot \mathbf{r}_l} e^{i\mathbf{q} \cdot \mathbf{r}_m} e^{i\mathbf{k} \cdot \mathbf{r}_n} \right| \Omega \right\rangle \\ = \frac{1}{N} \sum_{l \neq m \neq n} \langle \Omega | e^{-i\mathbf{k} \cdot \mathbf{r}_l} \left[ \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} e^{-i\mathbf{q} \cdot \mathbf{r}_l} e^{i\mathbf{q} \cdot \mathbf{r}_m} \right] e^{i\mathbf{k} \cdot \mathbf{r}_n} | \Omega \rangle \quad . \quad (21)$$

The sum over  $q$  is simply a delta function, so that

$$\frac{1}{\mathcal{A}} \sum_{\mathbf{q}} h^{(3)}(\mathbf{k}, \mathbf{q}) = \frac{1}{N} \sum_{l \neq m \neq n} \langle \Omega | e^{-i\mathbf{k} \cdot \mathbf{r}_l} \delta^2(\mathbf{r}_l - \mathbf{r}_m) e^{i\mathbf{k} \cdot \mathbf{r}_n} | \Omega \rangle \quad (22)$$

but, as seen above, this must be identically zero, the only configurations for which the delta-function is non-zero are ones for which the ground state wave function vanishes by virtue of the Pauli exclusion principle. Hence  $\frac{1}{\mathcal{A}} \sum_{\mathbf{q}} h^{(3)}(\mathbf{k}, \mathbf{q}) = 0 \quad \forall \mathbf{k}$ .

The remaining terms are simplified by noting that

$$e^{-\frac{1}{2}k^2} + s(k) - 1 = \overline{s}(k) \quad . \quad (23)$$

and by removing possible singularities by writing

$$s(q) - 1 = h(q) + N\delta_{\mathbf{q},\mathbf{0}} \quad (24)$$

where

$$h(q) = \rho_0 \int (g(r) - 1) e^{i\mathbf{q}\cdot\mathbf{r}} d^2\mathbf{r} \quad (25)$$

and  $g(r)$  is the usual radial distribution function of the electron liquid. Hence we are left with

$$\begin{aligned} \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} P_0(\mathbf{k}, \mathbf{q}) &= \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} e^{-\frac{1}{2}|\mathbf{q}|^2 - \frac{1}{2}\mathbf{k}^*\mathbf{q}\bar{s}(|\mathbf{k}|)} + 2\rho_0 e^{-\frac{1}{2}|\mathbf{k}|^2} \\ &+ \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} e^{-\frac{1}{2}(\mathbf{k}+\mathbf{q})^*\mathbf{k}h(|\mathbf{q}|)} + \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} e^{\frac{1}{2}\mathbf{q}^*\mathbf{k}h(|\mathbf{k}+\mathbf{q}|)} \\ &= \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} e^{-\frac{1}{2}|\mathbf{q}|^2 - \frac{1}{2}\mathbf{k}^*\mathbf{q}\bar{s}(|\mathbf{k}|)} + 2\rho_0 e^{-\frac{1}{2}|\mathbf{k}|^2} \end{aligned} \quad (26)$$

$$+ 2e^{-\frac{1}{2}|\mathbf{k}|^2} \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} e^{-\frac{1}{2}\mathbf{q}^*\mathbf{k}h(|\mathbf{q}|)} \quad . \quad (27)$$

As shown below in the appendix the summations can be carried out in the limit  $N, \mathcal{A} \rightarrow \infty$ ,  $N/\mathcal{A} = \rho_0$  to give

$$\frac{1}{\mathcal{A}} \sum_{\mathbf{q}} P(\mathbf{k}, \mathbf{q}) = \frac{\bar{s}(|\mathbf{k}|)}{2\pi} + 2\rho_0 e^{-\frac{1}{2}|\mathbf{k}|^2} + 2e^{-\frac{1}{2}|\mathbf{k}|^2} (-\rho_0) \quad (28)$$

$$= \frac{\bar{s}(k)}{2\pi} \quad . \quad (29)$$

which is our desired sum-rule.

We can use the symmetry properties derived above to derive a second version of this. Consider

$$\begin{aligned} \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} (P(\mathbf{k}, \mathbf{q}) - P(\mathbf{q}, \mathbf{k})) &= i \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} \Phi(\mathbf{q}, \mathbf{k}) \bar{s}(|\mathbf{k} + \mathbf{q}|) \\ &= i \frac{1}{\mathcal{A}} \sum_{\mathbf{q}} \Phi(\mathbf{q} - \mathbf{k}, \mathbf{k}) \bar{s}(q) \quad . \end{aligned} \quad (30)$$

It is shown in the appendix that this integral vanishes so that

$$\frac{1}{\mathcal{A}} \sum_{\mathbf{k}} P(\mathbf{k}, \mathbf{q}) = \frac{\bar{s}(q)}{2\pi} \quad . \quad (31)$$

Finally the reality of the right-hand side of 28 allows a further form of the sum-rule to be written as

$$\frac{1}{\mathcal{A}} \sum_{\mathbf{q}} P(\mathbf{k} + \mathbf{q}, -\mathbf{q}) = \frac{\bar{s}(k)}{2\pi} \quad . \quad (32)$$

The basic idea used here could be extended to consider sum-rules for higher order correlation functions were they to be of interest.



## 5 Assessment of the Convolution Approximation

In their work MacDonald et al. [11] estimated the small  $q$  behaviour of  $P(\mathbf{k}, \mathbf{q})$  by using the convolution approximation[12] for the 3-particle distribution function

$$n^{(3)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'') = \sum_{l \neq m \neq n} \langle \Omega | \delta(\mathbf{r} - \mathbf{r}_l) \delta(\mathbf{r}' - \mathbf{r}_m) \delta(\mathbf{r}'' - \mathbf{r}_n) | \Omega \rangle$$

which can be written as

$$\begin{aligned} n_c^{(3)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'') &= \rho^3 \{1 + h(r - r') + h(r' - r'') + h(r'' - r)\} \\ &+ \rho^3 \{h(r - r') h(r' - r'') + h(r' - r'') h(r'' - r) + h(r'' - r) h(r - r')\} \\ &+ \rho^4 \int h(r - R) h(r' - R) h(r'' - R) d^2 \mathbf{R} \quad . \end{aligned}$$

The three-point function,  $h^{(3)}(\mathbf{k}, \mathbf{q})$ , can be written as

$$h^{(3)}(\mathbf{k}, \mathbf{q}) = \frac{1}{N} \int d^2 \mathbf{r} d^2 \mathbf{r}' d^2 \mathbf{r}'' e^{-i(\mathbf{k} + \mathbf{q}) \cdot \mathbf{r}} e^{i\mathbf{q} \cdot \mathbf{r}'} e^{i\mathbf{k} \cdot \mathbf{r}''} n_c^{(3)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'')$$

which gives, using the convolution approximation

$$\begin{aligned} h_c^{(3)}(\mathbf{k}, \mathbf{q}) &= N^2 \delta_{\mathbf{k}, \mathbf{0}} \delta_{\mathbf{q}, \mathbf{0}} + N \delta_{\mathbf{k}, \mathbf{0}} h(q) + N \delta_{\mathbf{q}, \mathbf{0}} h(k) + N \delta_{\mathbf{k} + \mathbf{q}, \mathbf{0}} h(k) \\ &+ h(k) h(q) + [h(k) + h(q) + h(k) h(q)] h(|\mathbf{k} + \mathbf{q}|) \quad . \end{aligned}$$

This approximation improves on the standard Kirkwood decoupling in that it correctly gives the  $q \rightarrow 0$  limit as

$$h_c^{(3)}(\mathbf{k}, \mathbf{0}) = (N - 2)(s(k) - 1) .$$

Since  $P_0(\mathbf{k}, \mathbf{q})$  saturates our sum-rule we expect that

$$\frac{1}{A} \sum_{\mathbf{q}} h^{(3)}(\mathbf{k}, \mathbf{q}) = 0 \quad .$$

Using the convolution form gives

$$\begin{aligned} \frac{1}{A} \sum_{\mathbf{q}} h_c^{(3)}(\mathbf{k}, \mathbf{q}) &= \rho N \delta_{\mathbf{k}, \mathbf{0}} + N \delta_{\mathbf{k}, \mathbf{0}} \frac{1}{A} \sum_{\mathbf{q}} h(q) + 2\rho h(k) \\ &+ \frac{1}{A} \sum_{\mathbf{q}} \{h(k) h(q) + (h(k) + h(q) + h(k) h(q)) h(|\mathbf{k} + \mathbf{q}|)\} \end{aligned}$$

Now

$$\begin{aligned} \frac{1}{A} \sum_{\mathbf{q}} h(q) &\rightarrow \rho \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \int d^2 \mathbf{r} e^{-\mathbf{q} \cdot \mathbf{r}} (g(r) - 1) \\ &= -\rho \end{aligned}$$

so that

$$\begin{aligned} F(k) &= \frac{1}{A} \sum_{\mathbf{q}} h_c^{(3)}(\mathbf{k}, \mathbf{q}) = (1 + h(k)) \frac{1}{A} \sum_{\mathbf{q}} h(q) h(|\mathbf{k} + \mathbf{q}|) \\ &\rightarrow (1 + h(k)) \int \frac{d^2 \mathbf{q}}{(2\pi)^2} h(q) h(|\mathbf{k} + \mathbf{q}|) \quad . \end{aligned}$$

This final integral requires a specific form for the pair correlation function,  $h(q)$ . For the primary fractional quantum Hall states ( $\nu = 1/m$ , for odd  $m$ ) this is known from Monte-Carlo studies of the Laughlin wave function [5] which lead to the form

$$h(q) = -\nu e^{-q^2/2} + 4\nu e^{-q^2} \sum_m c_m L_m(q^2)$$

where the  $c_m$  co-efficients are tabulated in [5]. We have estimated  $F(k)$  numerically for the case  $\nu = 1/3$  and it is plotted in figure 1, along with a plot of  $\bar{s}(k)/2\pi$  for comparison. Clearly  $F(k)$  is not identically zero and is not even negligible in comparison to  $\bar{s}(k)/2\pi$ . Hence we deduce that this approximation is deficient. MacDonald et al.[11] themselves pointed out that this approximation could well be unreliable as it does not correctly reflect the particle-hole symmetry ( $\nu \rightarrow 1 - \nu$ ) of the system.

## 6 Summary and Discussion

In this short note we have derived a sum rule (with symmetry related variants) for the static structure factor and the three-point correlation function that will determine the leading finite-temperature corrections to the absorption rates of phonons and photons in spectroscopic studies of the fractional quantum Hall effect. In principle the approach followed would allow the formulation of similar sum-rules for higher order correlation functions should such objects ever become relevant to experimental work. The sum-rule for the three-point function has been used to assess the validity of the convolution approximation [12] for fractional quantum Hall systems and has shown that it does not capture all of the physics.

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## A Evaluation of Integrals

First of all we need to evaluate the following integral for later use:

$$\int_0^{2\pi} e^{-ze^{i\phi}} d\phi = \int_0^{2\pi} e^{-z \cos \phi} \{\cos(z \sin \phi) - i \sin(z \sin \phi)\} d\phi \quad (33)$$

now [13]

$$\begin{aligned}\int_0^{2\pi} e^{-z \cos \phi} \cos(z \sin \phi) d\phi &= 2\pi \\ \int_0^{2\pi} e^{-z \cos \phi} \sin(z \sin \phi) d\phi &= 0\end{aligned}$$

so that

$$\int_0^{2\pi} e^{-ze^{i\phi}} d\phi = 2\pi \quad \forall z \quad . \quad (34)$$

Now we need to evaluate

$$f_1(k) = \frac{1}{\mathcal{A}} \sum_q e^{-\frac{1}{2}|q|^2 - \frac{1}{2}k^*q} \quad (35)$$

in the limit  $\mathcal{A} \rightarrow \infty$ ,  $N \rightarrow \infty$ ,  $N/\mathcal{A} = \rho_0$  this becomes

$$\frac{1}{4\pi^2} \int_0^\infty dq q e^{-q^2/2} \int_0^{2\pi} d\phi e^{-\frac{1}{2}kq e^{i\phi}} = \frac{1}{2\pi} \quad . \quad (36)$$

Similarly

$$\begin{aligned}f_2(k) &= \frac{1}{\mathcal{A}} \sum_q e^{-q^*k/2} h(|q|) \\ &\rightarrow \frac{1}{4\pi^2} \int_0^\infty dq q h(q) \int_0^{2\pi} d\phi e^{-\frac{1}{2}kq e^{i\phi}} \\ &= \frac{1}{2\pi} \int_0^\infty dq q h(q) \quad .\end{aligned} \quad (37)$$

The final integral can be evaluated quite generally.

$$\begin{aligned}\int_0^\infty dq q h(q) &= \frac{1}{2\pi} \int d^2\mathbf{q} h(q) \\ &= 2\pi\rho_0 h(0) \quad .\end{aligned} \quad (38)$$

Now for any spin-polarized fermi system  $g(r) = 1 + h(r) \rightarrow 0$  as  $r \rightarrow 0$  as a consequence of the Pauli principle, hence  $h(0) = -1$  and

$$\frac{1}{2\pi} \int_0^\infty dq q h(q) = -\rho_0 \quad . \quad (39)$$

Finally we need to evaluate

$$\frac{1}{\mathcal{A}} \sum_{\mathbf{q}} \Phi(\mathbf{q} - \mathbf{k}, \mathbf{k}) \bar{s}(q) \rightarrow \frac{1}{4\pi^2} e^{-k^2/2} \int_0^\infty dq q \bar{s}(q) \int_0^{2\pi} d\phi 2e^{\frac{1}{2}kq \cos \phi} \sin\left(\frac{1}{2}kq \sin \phi\right) \quad . \quad (40)$$

As we have already seen the  $\phi$  integral vanishes and so, therefore does the whole expression.

## References

- [1] C.J. Mellor, R.H. Eyles, J.E. Digby, A.J. Kent, K.A. Benedict, L.J. Challis, M. Henini, C.T. Foxon, J.J. Harris, Phys. Rev. Lett., **74**, 2339 (1995).
- [2] U. Zeitler, A.M. Devitt, J.E. Digby, C.J. Mellor, A.J. Kent, K.A. Benedict, T. Cheng, Phys. Rev. Lett., **82**, 5333 (1999).
- [3] A. Pinczuk, B. Dennis, L.N. Pfeiffer, K. West, Phys. Rev. Lett., **70**, 3983 (1993).
- [4] A. Pinczuk, B. Dennis, A.S. Plaut, L.N. Pfeiffer, K. West, Proceedings of the 12th International Conference on High Magnetic Fields in Semiconductors, p83 (Singapore: World Scientific) (1997).
- [5] S.M. Girvin, A.H. MacDonald, P.M. Platzman, Phys. Rev. **B33**, 2481 (1986).
- [6] R.B. Laughlin, Phys. Rev. Lett., **50**, 1395 (1983).
- [7] P.M. Platzman, S. He, Phys. Rev. **B49**, 13674 (1994).
- [8] K.A. Benedict, R.K. Hills, C.J. Mellor, Phys. Rev. B **60**, 10984 (1999).
- [9] A.H. MacDonald, in Course 12 of *Mesoscopic Quantum Physics: Proceedings of the 61st Les Houches Summer School*, ed E. Akkermans, G. Montambaux, JL Pichard and J. Zinn-Justin, (Amsterdam: Elsevier) (1994).
- [10] S.M. Girvin and T. Jach, Phys. Rev. **B29**, 5617 (1984).
- [11] A.H. MacDonald, K.L. Liu, S.M. Girvin, P.M. Platzman, Phys. Rev. **B33**, 4014 (1986).
- [12] H.W. Jackson and E. Feenberg, Rev. Mod. Phys. **34**, 686 (1962).
- [13] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, formulae 3.931.4 and 3.931.2, 4th edition ed. and trans. by A. Jeffreys (New York: Academic Press) (1980).

## B Figure Caption

Figure 1: A plot of the function  $F(k)$  estimated numerically (full line) with a plot of  $\bar{s}(k)/2\pi$  plotted (dashed line) for comparison.

